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CALCULATION OF FIELDS IN LAMINATED STRUCTURES AND COMPUTATION OF INTEGRAL PARAMETERS

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The author presents a calculation of fields in laminated structures and a computation of integral parameters by a method suggested in the present work and used for mixtures with spherical particles. Based on Lorenz' determination of mean quantities, a new model of a disperse system (DS) is suggested. Using the theorem of a vector field, the fields and mean values of inhomogeneous systems are calculated from the condition of equality between the energies of the electric fields of the real medium and its model. The method is more general and permits one to obtain new formulas for calculating mean parameters of DS.

The study of potential fields inhomogeneous structures has long attracted the attention of scientists. The appearance of a multitude of theories, methods, and models for calculating the dielectric permeability ε (in the general case, the generalized conductivity) of mixtures was motivated by the difficulty of calculating fields in heterogeneous media. Investigators attempted, and are now trying, to overcome these difficulties by using various methods that lead to different determinations [1–22].

In the Maxwell-Lorenz theory, the dielectric permeability (DP) of a disperse system is determined from the condition of equality between the mean polarizations of a homogeneous medium and inclusions; using various models, a familiar formula was derived for calculating ε [1, 2]. In Wagner's theory the same result was obtained from the condition of equality between the potentials of an equally large homogeneous medium and inclusions [5]. Other investigators also arrived at similar results by different methods [4, 11, 12]. In [9] a new result was obtained by simplifying Lorenz's model.

The Rayleigh theory was based on Maxwell's idea about the possibility of representing the solutions of Laplace equations in and outside inclusions in the form of potentials of multifields [3]; the theory of multifields was also used in [10].

In the theory of calculating the DP for an inhomogeneous medium Wiener was the first to determine ε from the ratio of the mean values of the electric displacement D and the electric field strength E [4]. Bruggeman proposed a new technique to determine ε on the basis of the integral method of calculating the dielectric permeability of a disperse system [8]. Disperse systems with differently shaped inclusions were considered in [6, 7]. A survey and comparison of different theories, models, and methods of calculation of fields and computation of integral parameters of DS are presented in greater detail in [12, 14–16].

Analysis of the proposed theories and formulas derived for calculating ε shows that they can be divided into two basic groups. The theories and formulas of one group are based on certain model representations of disperse systems, while in the theories and formulas of the second type a correction is sought for the mean value of integral parameters of mixtures.

The determination of the dielectric permeability according to Wiener or Lorenz means that a real disperse system is replaced by an equivalent homogeneous medium with the unknown value of ε. From the energy point of view this means that the mixture and the homogeneous medium should posssess the same value of energy of electric (or other) fields of these systems. But the energy, including electrical, represents a more general notion than any other quantity characterizing one or another medium [23]. Proceeding from the above, in [17–21] a method was suggested in which the dielectric permeability of a disperse system was determined from the condition of equality

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between the energies of a disperse system and a homogeneous medium. This determination is more general and is applicable for any models of disperse systems and any shape of the inclusions of the disperse phase. In the present paper a calculation of the fields in laminated structures and a computation of mean parameters by the method used in [17–22] are performed.

1. Laminated Two-Component Structures. A simple model of two-component media is the concept of parallel consecutive incorporation of its components. In the case of a model of a two-component medium with consecutive incorporation of its components, the interface between two components is normal to the vector of the external field of strength $\mathbf{E}_0$ [12]. For a model with parallel incorporation of its components, the interface between two components is parallel to the direction of the external field.

To calculate the field and compute the integral value of the dielectric permeability of a DS, disperse systems with spherical inclusions were considered in [21]. Using the definition of the mean value of the DP of a mixture according to the expression

$$\varepsilon_{\mathrm{m}} = \varepsilon_{\mathrm{mix}} = \frac{1}{V_0} \int_{V_0} \varepsilon_{\mathrm{mic}}\, dV, \tag{1}$$

we can represent a real disperse system by an equivalent model in which the space of a DS of volume V_0 is replaced by a homogeneous medium with DP $\varepsilon_{\mathrm{mix}}$. For the real medium and its model to be equivalent, it is sufficient that outside the volume V_0, within which averaging of (1) is performed, the potentials of the field at the same points be equal in both systems. Then, from the condition of equality between the energy of the averaged homogeneous body W_{mix} of volume V_0 with DP $\varepsilon_{\mathrm{mix}}$ and the energy of the real medium, occupying the volume V_0, with n_k foreign particles W_{p} [21], we obtain

$$W_{\mathrm{mix}} = \frac{1}{2} (\varepsilon_e - \varepsilon_{\mathrm{mix}}) \int_{V_0} \mathbf{E}_0 \mathbf{E}_{\mathrm{mix}\, i}\, dV = \frac{1}{2} (\varepsilon_e - \varepsilon_i) \sum_{k=1}^{n_k} \int_{V_{ki}} \mathbf{E}_0 \mathbf{E}_{ki} dV = W_{\mathrm{p}}, \tag{2}$$

where ε_e is the DP of the disperse medium, ε_i is the DP of the i-th particle of the disperse phase; $\mathbf{E}_0$ is the strength of the external homogeneous electrostatic field; $\mathbf{E}_{\mathrm{mix}\, i}$ is the strength within the equivalent homogeneous body; $\mathbf{E}_{ki}$ is the strength within the k-th foreign particle; V_{ki} is the volume of the k-th particle.

Taking into account Eq. (2), we introduce the first layer of volume V_1 (the model with consecutive incorporation of layers) and DP ε_1 into a dispersion medium with DP ε_1, in which a homogeneous field of strength $\mathbf{E}_0$ is established. According to Eq. (2), the energy of this layer $W_{\mathrm{p}1}$ in the field $\mathbf{E}_0$ is equal to zero. And for the energy of the second layer with DP ε_2 and volume V_2 we have

$$W_{\mathrm{p}_2} = \frac{1}{2} (\varepsilon_1 - \varepsilon_2)\, \mathbf{E}_0 \int_{V_2} \mathbf{E}_{2i}\, dV. \tag{3}$$

An expression similar to Eq. (3) can be obtained if the model with consecutive incorporation of layers is introduced simultaneously into the dispersion medium.

We will introduce a homogeneous body of volume $V = V_1 + V_2$ and DP $\varepsilon_{12} = \varepsilon_{\mathrm{mix}}$ into the dispersion medium (the model is presented by a homogeneous medium). For the energy $W_{12} = W_{\mathrm{mix}}$ of this homogeneous body placed in the field $\mathbf{E}_0$ we obtain

$$W_{\mathrm{mix}} = \frac{1}{2} (\varepsilon_1 - \varepsilon_{12})\, \mathbf{E}_0 \int_{V} \mathbf{E}_i dV. \tag{4}$$

According to [21], the energies (3) and (4) are equal:

$$\frac{1}{2} (\varepsilon_1 - \varepsilon_2)\, \mathbf{E}_0 \int_{V_2} \mathbf{E}_{2i} dV = \frac{1}{2} (\varepsilon_1 - \varepsilon_{12})\, \mathbf{E}_0 \int_{V} \mathbf{E}_i dV. \tag{5}$$

Upon introduction of the second layer into the dispersion medium we write the following condition on the interface:

$$\varepsilon_2 \mathbf{E}_{2i} = \varepsilon_1 \mathbf{E}_0 . \tag{6}$$

And upon introduction of a homogeneous body into the dispersion medium we have on the interface that

$$\varepsilon \mathbf{E}_i = \varepsilon_1 \mathbf{E}_0 . \tag{6a}$$

Consequently,

$$\mathbf{E}_{2i} = \frac{\varepsilon}{\varepsilon_2} \mathbf{E}_i . \tag{6b}$$

Substituting (6b) into Eq. (5), for $\varepsilon_{12} = \varepsilon_{\mathrm{mix}}$ we obtain the well-known Maxwell formula

$$\varepsilon_{\mathrm{mix}} = \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_2 + f_2 (\varepsilon_1 - \varepsilon_2)} , \tag{7}$$

where $f_2 = V_2 / V$.

In deriving Eq. (7) it was taken into account that

$$\mathbf{E}_1 d_1 + \mathbf{E}_2 d_2 = \mathbf{E}_0 d , \tag{7a}$$

where d_1, d_2, d are the thicknesses of the first and second layer and the homogeneous body, respectively.

We must infer that the derivation of Eq. (7) should not depend on the dielectric permeability of teh dispersion medium into which the layers of the model under consideration and the homogeneous body are introduced. We can demonstrate this by assuming the dielectric permeability of the dispersion medium to be equal to the arbitrary value ε_x. Then, in accordance with Eq. (2), the equality of energies (5) will take the form

$$\mathbf{E}_0 (\varepsilon_x - \varepsilon_1) V_1 \mathbf{E}_{1i} + \mathbf{E}_0 (\varepsilon_x - \varepsilon_2) V_2 \mathbf{E}_{2i} = (\varepsilon_x - \varepsilon_{\mathrm{mix}}) V \mathbf{E}_0 \mathbf{E}_i . \tag{8}$$

Taking into account the boundary conditions for the laminated structure

$$\varepsilon_1 \mathbf{E}_{1i} = \varepsilon_2 \mathbf{E}_{2i} = \varepsilon_x \mathbf{E}_0 \tag{9}$$

and for the model

$$\varepsilon_{\mathrm{mix}} \mathbf{E}_i = \varepsilon_x \mathbf{E}_0 , \tag{10}$$

and also the fact that $f_1 = V_1 / V = 1 - f_2$, we again obtain Eq. (7) from Eq. (8). This was to be expected because, if we assume that $\varepsilon_1 = 1$ (or $\varepsilon_x = 1$), the equalities of the energies of the dielectric bodies (3) and (4), and consequently also (5) or (8), transform to equalities of the polarization energies of these bodies. And, as is known, the polarization energy of a dielectric substance depends on the properties (DP) of the substance and the external field applied. It should also be emphasized that the expression for the energy of a dielectric body (2) (W_{mix} and W_{p}) placed in an external electrostatic field was derived from the difference between the energies of the fields of the disperse system and the dispersion medium.

For the model with parallel layers Eq. (5) assumes the form

$$(\varepsilon_1 - \varepsilon_2) V_2 = (\varepsilon_1 - \varepsilon_{\mathrm{mix}}) V , \tag{11}$$

since $\mathbf{E}_{1i} = \mathbf{E}_{2i} = \mathbf{E}_i = \mathbf{E}_0$ in this case. For the case considered this equation gives the well-known formula

$$\varepsilon_{\mathrm{mix}} = \varepsilon_2 + f_2 (\varepsilon_1 - \varepsilon_2) . \tag{12}$$

2. Disperse Systems with Laminated Spherical Inclusions. First, we consider the case of a single two-layer spherical particle placed in an external uniform electrostatic field of strength E_0. Suppose the internal layer of radius r_1 with dielecytic permeability ε_1 is covered by a layer with DP ε_2. The external radius of the particle is equal to r_2. It is necessary to calculate the equivalent DP of the particle $\varepsilon_{12} = \varepsilon_{mix}$. First, we introduce a laminated spherical particle into a dispersion medium with DP $\varepsilon_2 = 1$, in which the field E_0 is established. The polarization energy of this particle in the field E_0 is equal to the sum of the polarization energies of the internal W_{p1} and external W_{p2} layers. And since, according to Eq. (2) or (3), W_{p2} is equal to zero, then

$$W_{p_{12}} = W_{p_1} = \frac{1}{2}(1 - \varepsilon_1)\,\varepsilon_0 E_0 \int_{V_1} E_{1i} dV. \tag{13}$$

Into such a dispersion medium, we introduce a homogeneous sphere of radius r_2 and DP ε_{12}. The polarization energy of this body placed in the field E_0 will be equal to

$$W_{p_{12}} = \frac{1}{2}(1 - \varepsilon_{12})\,\varepsilon_0 E_0 \int_{V_{12}} E_{12i}\, dV. \tag{14}$$

The equality of (13) and (14)

$$\frac{1}{2}(1 - \varepsilon_1)\,\varepsilon_0 E_0 \int_{V_1} E_{1i} dV = \frac{1}{2}(1 - \varepsilon_{12})\,\varepsilon_0 E_0 \int_{V_{12}} E_{12i}\, dV \tag{15}$$

permits one to calculate ε_{12}. As shown above, equality (15) is independent of the dielectric permeability of the dispersion medium.

The calculation of E_{1i} and E_{2i} presents no difficulty (the procedure is well known and is cited in every textbook on theoretical electrical engineering).

Expressing W_{p1} (15) in terms of the dipole moment p_1 of a particle of radius r_1

$$W_{p_1} = \frac{1}{2}(1 - \varepsilon_1)\,\varepsilon_0 E_0 \int_{V_1} E_{1i} dV = -\frac{1}{2} E_0 \int_{V_1} \frac{p_1}{V_1}\, dV \tag{16}$$

and taking into account that inside the particle the field strength is equal to the sum of the field E_0 and the polarization field E_p

$$E_{1i} = E_0 + E_p = E_0 - \frac{P_1}{3\varepsilon_0} = E_0 - \frac{p_1}{3\varepsilon_0 V_1}, \tag{17}$$

by solving Eqs. (16) and (17) simultaneously we obtain for E_{1i} at $\varepsilon_2 \neq 1$

$$E_{1i} = \frac{3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} E_0. \tag{18}$$

Following the same lines we can calculate the field inside a homogeneous sphere with DP ε_{12} (at $\varepsilon_2 \neq 1$):

$$E_{12i} = \frac{3\varepsilon_2}{\varepsilon_{12} + 2\varepsilon_2} E_0. \tag{19}$$

Substituting Eqs. (18) and (19) into Eq. (15), for the equivalent permeability we obtain a formula derived by Netushil:

$$\varepsilon_{12} = \varepsilon_2 \frac{\varepsilon_1 (r_2^3 + 2r_1^3) + 2\varepsilon_2 (r_2^3 - r_1^3)}{\varepsilon_2 (2r_2^3 + r_1^3) + \varepsilon_1 (r_2^3 - r_1^3)} . \tag{20}$$

We note that Eq. (20) is Maxwell's formula in a different formulation

$$\varepsilon_{12} = \varepsilon_2 \frac{\varepsilon_1 + 2\varepsilon_2 + 2f_1 (\varepsilon_1 - \varepsilon_2)}{\varepsilon_1 + 2\varepsilon_2 - f_1 (\varepsilon_1 - \varepsilon_2)} , \tag{21}$$

where $f_1 = V_1/V_{12} = r_1^2/r_2^3 = (1 - h/r_2)^3$; h is the thickness of the external layer.

Let us consider a new problem: in a dispersion medium the indicated laminated spherical particles are located regularly at the nodes of a simple cubic lattice. First, we consider the case where the volumetric fraction is insignificant and the interaction of particles is neglected.

We begin the calculation of the field and the computation of the DP of the mixture ε_{mix} with the replacement of the laminated particles by equivalent particles with DP ε_{12} of (20). Applying the Lorenz averaging (1) to such a medium in the spherical volume V_0 (of radius R_0), we represent the disperse system by a homogeneous sphere of volume V_0 with the unknown DP ε_{mix} [21]. Then, in this case relation (5) will take the form

$$\frac{1}{2} (1 - \varepsilon_{12}) \varepsilon_0 \mathbf{E}_0 \sum_{i=1}^{n} \int_{V_{12}} \mathbf{E}_{12i} \, dV = \frac{1}{2} (1 - \varepsilon_{mix}) \varepsilon_0 \mathbf{E}_0 \int_{V_0} \mathbf{E}_{mix\, i} \, dV , \tag{22}$$

where n is the number of particles contained in the spherical volume V_0 of radius R_0; $\mathbf{E}_{12i}$ is the strength inside a homogeneous particle of radius r_2 and DP ε_{12}; $\mathbf{E}_{mix\, i}$ is the strength inside the equivalent homogeneous sphere of radius R_0 and DP ε_{mix}.

At low volumetric concentrations of the inclusions $f_{12} = nV_{12}/V_0 = nV_2/V_0$ the particles are polarized by the external field $\mathbf{E}_0$, and in Eq. (22) $\mathbf{E}_{12i}$ is determined from Eq. (19) with the replacement of ε_2 by $\varepsilon_2 = 1$. Then, taking account of the symmetry we have from Eq. (22) that

$$\frac{\varepsilon_{mix} - 1}{\varepsilon_{mix} + 2} = f_{12} \frac{\varepsilon_{12} - 1}{\varepsilon_{12} + 2} , \tag{23}$$

where ε_{12} is determined from Eq. (20). We note that relation (23) is known as the Maxwell–Wagner formula for disperse systems with laminated spherical inclusions.

But if the volumetric fraction of the laminated particles f_{12} is such that it is necessary to take into account dipole interactions, the particles will be polarized by the Lorenz field. In this case

$$\mathbf{E}_{12i} = \mathbf{E}_L + \mathbf{E}_p = \mathbf{E}_0 + \frac{\mathbf{P}_{12}}{3\varepsilon_0} - \frac{\mathbf{p}_{12}}{3\varepsilon_0 V_{12}} = \mathbf{E}_0 + \frac{n\mathbf{p}_{12}}{3\varepsilon_0 V_0} - \frac{\mathbf{p}_{12}}{3\varepsilon_0 V_{12}} . \tag{24}$$

In Eq. (24) the averaging of the dipole moments $\mathbf{p}_{12}$ of the particles is performed in the volume V_0.

Expressing the energy spent for polarization of the particles (14) in terms of the dipole moment $\mathbf{p}_{12}$ and solving the thus obtained equation simultaneously with Eq. (24), we obtain by analogy with Eqs. (16) and (17) that

$$\mathbf{E}_{12i} = \frac{3\mathbf{E}_0}{\varepsilon_{12} + 2 - f_{12} (\varepsilon_{12} - 1)} , \tag{25}$$

where $f_{12} = nV_{12}/V_0$.

Now, substituting Eq. (25) into Eq. (22), with allowance for dipole interactions of the particles, we obtain a new formula for the dielectric permeability of the mixture:

$$\frac{\varepsilon_{mix} - 1}{\varepsilon_{mix} + 2} = f_{12} \frac{\varepsilon_{12} - 1}{\varepsilon_{12} + 2 - f_{12}(\varepsilon_{12} - 1)}. \tag{26}$$

We must infer that the accuracy of this formula should be higher than that of Eq. (23), since in contrast to the latter, it takes into account the dipole interaction.

To calculate ε_{12} taking account of the dipole interaction we substitute the value of the Lorenz field in Eq. (17) instead of $\mathbf{E}_0$ (in Eqs. (16) and (24) $\mathbf{p}_{12}$ is replaced by $\mathbf{p}_2$) and solve it simultaneously with Eq. (16). For $\mathbf{E}_{1i}$ we obtain (at $\varepsilon \neq 1$)

$$\mathbf{E}_{1i} = \frac{3\varepsilon_2 \mathbf{E}_0}{\varepsilon_1 + 2\varepsilon_2 - f_{12}(\varepsilon_1 - \varepsilon_2)}. \tag{27}$$

Then from Eq. (15), taking into account Eqs. (25) and (27), we obtain

$$\frac{\varepsilon_{12} - \varepsilon_2}{\varepsilon_{12} + 2\varepsilon_2 - f_{12}(\varepsilon_{12} - \varepsilon_2)} = f_1 \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2 - f_{12}(\varepsilon_1 - \varepsilon_2)}. \tag{28}$$

If we neglect the dipole interaction, then we obtain formula (20) from Eq. (28).

We may claim that the accuracy of the new formulas (26) and (28), which take into account the dipole interaction, is higher than the accuracy of the well known formulas (20) and (23). This is confirmed by the fact that our formula (26) for ordinary spherical inclusions correlates better with experiment than the Maxwell-Lorenz formula (23) [21, 22].

It should be noted that the method suggested in [21] permits one to calculate the field and compute integral parameters of a disperse system with laminated spherical particles with allowance for the interaction of higher-order multifields.

3. Disperse Systems with Laminated Cylindrical Inclusions. Suppose there is a rather long cylinder of radius r_1 with DP ε_1 covered by a layer of another material of thickness h with DP ε_2. The radius of the external cylinder is $r_2 = r_1 + h$. The axis of the cylinder is perpendicular to the direction of an external electrostatic field with the strength $\mathbf{E}_0$. Performing calculations similar to those for a laminated sphere and taking into account the fact that the depolarization factor along the X axis is $N_x = 1/2$, we obtain for the strength $\mathbf{E}_{1ix}$ inside the layer with DP ε_1 and the strength $\mathbf{E}_{12ix}$ inside the homogeneous cylinder with DP ε_{12} instead of Eqs. (18) and (19) ($\varepsilon_2 \neq 1$)

$$\mathbf{E}_{1ix} = \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \mathbf{E}_{0x}, \tag{29}$$

$$\mathbf{E}_{12ix} = \frac{2\varepsilon_2}{\varepsilon_{12} + \varepsilon_2} \mathbf{E}_{0x}. \tag{30}$$

Substituting Eqs. (23) and (30) into Eq. (15) (at $\varepsilon_2 \neq 1$), for the dielectric permeability of the laminated cylinder we obtain

$$\frac{\varepsilon_{12} - \varepsilon_2}{\varepsilon_{12} + \varepsilon_2} = f_1 \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}, \tag{31}$$

whence

$$\varepsilon_{12} = \varepsilon_2 \frac{\varepsilon_1 (r_2^2 + r_1^2) + \varepsilon_2 (r_2^2 - r_1^2)}{\varepsilon_2 (r_2^2 + r_1^2) + \varepsilon_1 (r_2^2 - r_1^2)}, \tag{32}$$

where $f_1 = r_1^2/r_2^2$.

Let us calculate the field within laminated cylindrical particles whose axes coincide with the nodes of a spatial cubic lattice and are perpendicular to the direction of the external field. First, we consider the case where the volumetric fraction of the particles is very small and the mutual effect of the particles can be neglected.

Since the shape of the volume V_0, in which the averaging is performed, must be similar to the shape of the inclusions, this volume has the shape of a cylinder. In the case considered the particles are polarized by the external field, and $\mathbf{E}_{1ix}$ and $\mathbf{E}_{2ix}$ are determined from Eqs. (29) and (30). And the strength $\mathbf{E}_{\text{mix}\, ix}$ inside a homogeneous cylinder with DP ε_{mix} and volume V_0 is determined from Eq. (29) and is equal to (at $\varepsilon = 1$)

$$\mathbf{E}_{\text{mix}\, ix} = \frac{2\mathbf{E}_{0x}}{\varepsilon_{\text{mix}} + 1}. \tag{33}$$

From the energy balance (22), taking account of Eqs. (30) and (33), we have for the dielectric permeability of the mixture at $\varepsilon_2 \neq 1$

$$\varepsilon_{\text{mix}} = \varepsilon_2 \frac{\varepsilon_{12} + \varepsilon_2 + f_{12}(\varepsilon_{12} - \varepsilon_2)}{\varepsilon_{12} + \varepsilon_2 - f_{12}(\varepsilon_{12} - \varepsilon_2)}, \tag{34}$$

where $f_{12} = nV_2/V_0$.

In Eq. (34) ε_{12} is determined from Eq. (32). Formula (34) is similar to Rayleigh's formula for cylindrical inclusions [7]. If the volumetric fraction of the foreign bodies f_{12} is such that it is necessary to take into account the dipole interaction, then for the strength inside homogeneous cylinders with DP ε_{12} we have

$$\mathbf{E}_{12ix} = \mathbf{E}_{\text{L}x} + \mathbf{E}_{\text{p}x} = \mathbf{E}_{0x} + \frac{n\mathbf{p}_{12x}}{2\varepsilon_0 V_0} - \frac{\mathbf{p}_{12x}}{2\varepsilon_0 V_{12}}. \tag{35}$$

Performing analogous calculations for spherical particles, instead of Eq. (25) we can write

$$\mathbf{E}_{12ix} = \frac{2\mathbf{E}_{0x}}{\varepsilon_{12} + 1 - f_{12}(\varepsilon_{12} - 1)}. \tag{36}$$

Substituting Eqs. (33) and (36) into Eq. (22) at $\varepsilon_2 \neq 1$, we obtain a new formula:

$$\varepsilon_{\text{mix}} = \varepsilon_2 \frac{\varepsilon_{12} + \varepsilon_2}{\varepsilon_{12} + \varepsilon_2 - 2f_{12}(\varepsilon_{12} - \varepsilon_2)}. \tag{37}$$

Naturally, in calculating ε_{12} it is also necessary to take into account the interaction of the particles. Similar calculations for $\mathbf{E}_{1ix}$ yield

$$\mathbf{E}_{1ix} = \frac{2\mathbf{E}_{0x}}{\varepsilon_1 + 1 - f_{12}(\varepsilon_1 - 1)}. \tag{38}$$

Substituting Eqs. (36) and (38) into Eq. (15), for ε_{12} at $\varepsilon_2 \neq 1$ we obtain the formula

$$\varepsilon_{12} = \varepsilon_2 \frac{\varepsilon_1 + \varepsilon_2 - f_{12}(\varepsilon_1 - \varepsilon_2) + f_1(\varepsilon_1 - \varepsilon_2)(1 + f_{12})}{\varepsilon_1 + \varepsilon_2 - f_{12}(\varepsilon_1 - \varepsilon_2) - f_1(\varepsilon_1 - \varepsilon_2)(1 + f_{12})}. \tag{39}$$

4. Foreign Laminated Particles of Ellipsoidal Shape. Let an internal ellipsoid consist of a material with dielectric permeability ε_1 and an external layer consist of a material with permeability ε_2. The ellipsoidal surfaces of the layers are confocal. Let us assume that an external field is directed along the X axis, so that $\mathbf{E}_0 = \mathbf{E}_{0x}$, and the depolarization factor is N_x. To calculate the field within such a particle and compute the equivalent dielectric permeability ε_{12}, first, as in Secs. 2 and 3, we introduce an ellipsoid of volume V_1 with the permeability of the material ε_1 into a dispersion medium with DP ε_2. The strength of the field inside the ellipsoid along the X axis is equal to

$$\mathbf{E}_{1ix} = \mathbf{E}_{0x} + \mathbf{E}_{px} = \mathbf{E}_{0x} - \frac{N_x \mathbf{p}_{2x}}{\varepsilon_0 V_1}. \tag{40}$$

Performing calculations similar to those for spheres and cylinders, for the strength of the internal ellipsoid we obtain

$$\mathbf{E}_{1ix} = \frac{\mathbf{E}_{0x}}{1 + (\varepsilon_1 - 1)\, N_x}. \tag{41}$$

We now introduce a homogeneous ellipsoid of volume V_2 with DP ε_{12} into the same dispersion medium. For the strength inside this body we have

$$\mathbf{E}_{12x} = \frac{\mathbf{E}_{0x}}{1 + (\varepsilon_{12} - 1)\, N_x}. \tag{42}$$

Substituting Eqs. (41) and (42) into Eq. (15) at $\varepsilon_2 \neq 1$, we obtain the formula

$$\varepsilon_{12} = \varepsilon_2 \frac{\varepsilon_2 + (\varepsilon_1 - \varepsilon_2)\,(f_1 + (1 - f_1)\, N_x)}{\varepsilon_2 + (\varepsilon_1 - \varepsilon_2)\,(1 - f_1)\, N_x}, \tag{43}$$

which is similar to Fricke's formula [7, 14] for a disperse system with ordinary ellipsoids. Now let laminated identically oriented ellipsoids be located at the nodes of a spatial cubic lattice. At very low volumetric concentrations of the foreign bodies, the interaction can be neglected. Then, for the strength $\mathbf{E}_{\mathrm{mix}\, ix}$ within a homogeneous ellipsoid with DP $\varepsilon_{\mathrm{mix}}$ and volume V_0 we obtain

$$\mathbf{E}_{\mathrm{mix}\, ix} = \frac{\mathbf{E}_{0x}}{1 + (\varepsilon_{\mathrm{mix}} - 1)\, N_x}. \tag{44}$$

For the DP of a disperse system along the X axis (at $\varepsilon_2 \neq 1$) Eqs. (22), (42), and (44) permit one to obtain a new formula:

$$\varepsilon_{\mathrm{mix}} = \varepsilon_{12} \frac{\varepsilon_2 + (\varepsilon_{12} - \varepsilon_2)\,(f_{12} + (1 - f_{12})\, N_x)}{\varepsilon_2 + (\varepsilon_{12} - \varepsilon_2)\,(1 - f_{12})\, N_x}, \tag{45}$$

where $f_2 = V_2/V_0$, and ε_{12} is determined from Eq. (43).

Let us now go over to the calculation of the field and the computation of $\varepsilon_{\mathrm{mix}}$ with allowance for dipole interaction. For this, first we introduce n homogeneous identically oriented (along the X axis) ellipsoids with DP ε_{12} into a dispersion medium. The polarization energy of these particles (at $\varepsilon_2 = 1$) is equal to

$$W_{\mathrm{p}12} = \frac{1}{2}(1 - \varepsilon_{12})\, \varepsilon_0 \sum_{i=1}^{n} \int_{V_{12}} \mathbf{E}_{0x}\, \mathbf{E}_{12ix}\, dV. \tag{46}$$

In the present case these particles are polarized by the field

$$\mathbf{E}_{12ix} = \mathbf{E}_{\mathrm{L}x} + \mathbf{E}_{px} = \mathbf{E}_{0x} + \frac{nN_x\, \mathbf{p}_{12x}}{\varepsilon_0 V_0} - \frac{N_x\, \mathbf{p}_{12x}}{\varepsilon_0 V_{12}}. \tag{47}$$

Expressing Eq. (46) in terms of $\mathbf{p}_{12x}$ and solving it simultaneously with Eq. (47), we obtain for $\mathbf{E}_{12ix}$ the following relation:

$$E_{12ix} = \frac{E_{0x}}{1 - (\varepsilon_{12} - 1)(f_{12} - 1) N_x}. \tag{48}$$

Similarly, we find the strength of the field in the internal ellipsoids (with DP ε_1)

$$E_{1ix} = \frac{E_{0x}}{1 - (\varepsilon_1 - 1)(f_1 - 1) N_x}. \tag{49}$$

Substituting Eqs. (48) and (49) into Eq. (15), for ε_{12} at $\varepsilon_2 \neq 1$ we obtain

$$\varepsilon_{12} = \varepsilon_2 \frac{\varepsilon_2 + (\varepsilon_1 - \varepsilon_2)(f_1 + N_x (1 + 2f_1))}{\varepsilon_2 + (\varepsilon_1 - \varepsilon_2)(1 - 2f_{12}) N_x}, \tag{50}$$

where $f_1 = V_1 / V_{12}$.

Introducing into the dispersion medium a homogeneous ellipsoid with DP ε_{mix} of volume V_0, in which the averaging of $\mathbf{p}_{12}$ of (47) is performed we obtain the following relation for the polarization energy

$$W_{p\,mix} = \frac{1}{2}(1 - \varepsilon_{mix})\, \varepsilon_0 \int_{V_0} E_{0x} E_{mix\,ix}\, dV. \tag{51}$$

Equating (46) and (51) and taking into account Eqs. (48) and (49), for the dielectric permeability of the mixture (at $\varepsilon_2 \neq 1$) we have a new formula that is more accurate than formula (50):

$$\varepsilon_{mix} = \varepsilon_{12} \frac{\varepsilon_2 + (\varepsilon_2 - \varepsilon_{12})(f_{12} - N_x (1 + 2f_{12}))}{\varepsilon_2 + (\varepsilon_2 - \varepsilon_{12})(1 - 2f_{12}) N_x}. \tag{52}$$

We also obtain for ε_{12} a formula similar to formula (52)

$$\varepsilon_{12} = \varepsilon_2 \frac{\varepsilon_2 + (\varepsilon_2 - \varepsilon_1)(f_1 - N_x)(1 + 2f_1)}{\varepsilon_2 + (\varepsilon_2 - \varepsilon_1)(1 - 2f_1) N_x}. \tag{53}$$

In conclusion, we must note that the proposed method for calculating the field and computing integral parameters of a mixture with two-layer inclusions permits one to take into account more accurately the influence of a double layer on dielectric dispersion and also to calculate the density of bound charges on the surface of foreign particles.

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